23/3/2017
Μαθημα 8

Επαναληψη $f: A \to \mathbb{C}$ $\forall z = x+iy \in A$, $A \subseteq \mathbb{C}$
$f(z) = u(x,y) + iv(x,y)$ $\quad f(z) \in \mathbb{C}$
$f(z) = u + iv$, $\quad u = Re(f(z))$, $v = Im(f(z))$

Όριο: $\ell \in \mathbb{C}$, $z_0 \in A$, $\lim_{z \to z_0} f(z) = \ell : (\forall \varepsilon > 0)(\exists \delta > 0)$

$$f(B_0(z_0,\delta) \cap A) \subseteq B(\ell, \varepsilon)$$

$(\forall z \in A)$ $0 < |z - z_0| < \delta \Rightarrow |f(z) - \ell| < \varepsilon$

- $z_0 = x_0 + iy_0$ $\quad |z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$
- $z = x + iy$ $\quad f = u_0 + iv_0$

$$\sqrt{(u(x,y) - u_0)^2 + (v(x,y) - v_0)^2} < \varepsilon$$

$$\Rightarrow \begin{cases} |u(x,y) - u_0| < \varepsilon \\ |v(x,y) - v_0| < \varepsilon \end{cases}$$

$$\left.\begin{array}{l} x \to x_0 \\ y \to y_0 \\ \Rightarrow u(x,y) \to u_0 \\ \text{και } v(x,y) \to v_0 \end{array}\right\} \Longrightarrow \begin{array}{l} \exists \lim_{(x,y) \to (x_0,y_0)} u(x,y) \\ \exists \lim_{(x,y) \to (x_0,y_0)} v(x,y) \end{array}$$

Τελικα, $\lim_{z \to z_0} f(z) = \ell \Longrightarrow \lim_{z_0} Re(f(z)) = Re(\ell)$

$$\lim_{z_0} Im(f(z)) = Im(\ell)$$

Κανονες προσθεσης - γινομενου
$f, g: A \to \mathbb{C}$
$\exists \lim_{z_0} f$ και $\exists \lim_{z_0} g$

1) $\exists \lim_{z_0} (f+g) = \lim_{z_0} f + \lim_{z_0} g$

2) $\exists \lim_{z_0} (f \cdot g) = \lim_{z_0} f \cdot \lim_{z_0} g$

Ιδιοτητες:

1) $z \to z_0$ και $|z| \to |z_0|$

$||z|-|z_0|| \leq |z-z_0|$

2) $z \to z_0 \Rightarrow z^2 \to z_0^2$

$|z^2-z_0^2| = |(z-z_0)(z+z_0)| = |z-z_0| \cdot |z+z_0|$

$\leq |z-z_0|(|z|+|z_0|)$

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in \mathbb{C})\ |z-z_0| < \delta \Rightarrow |z^2-z_0^2| \leq \varepsilon$

$\left.\begin{array}{l} |z|-|z_0| \leq \delta \\ |z| \leq \delta + |z_0| \end{array}\right\} \ |z^2-z_0^2| \leq \delta(\delta + 2|z_0|) \leq \varepsilon$

Συνεχεια:

f συνεχης στο $z_0$: $\exists \lim_{z_0} f$ και

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \in A)\ |z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$

πχ: 1) $f(z) = z^2$

$\lim_{z_0} f = \lim_{z_0} z^2 = z_0^2 = f(z_0)$

2) $f(z) = \operatorname{Arg} z$, $z \neq 0$

ΔΕΝ ειναι συνεχης, οταν $z + |z| = 0$

$z = -|z|$, $\theta = -\pi$

παραδειγμα: Βρειτε τα σημεια του μιγαδικου επιπεδου που η $f(z) = \operatorname{Arg}(|z|^2 - iz^2)$ ειναι συνεχης.

Λυση: θετω $j = |z|^2 - iz^2$

Οχι συνεχης: $j + |j| = 0$ θετω $z = x+iy$

$j = x^2+y^2 - i(x^2-y^2+2ixy) = x^2+y^2+2xy - i(x^2-y^2)$

$j + |j| = 0 \Rightarrow (x+y)^2 - i(x^2-y^2) + |(x+y)^2| = 0$

$x^2 - y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$

$(x+y)^2 \mp |(x+y)^2| = 0$

$(x \pm x)^2 + (x \pm x)^2 = 0$

$y = -x$, $4x^2 + 4x^2 = 0 \Rightarrow x = 0,\ y = 0$

Δεν ειναι συνεχης στα σημεια οπου $y = -x$

$C(z, \mathbb{C})$

$f, g \in C(z, \mathbb{C}) \Rightarrow f+g,\ f \cdot g \in C(z, \mathbb{C})$

$z \in \gamma$

$z = z(t),\ t \in (a, \beta)$

$f(z) \ni \Gamma : f(z(t)) = w,\ t \in (a, \beta]$

$\gamma: z = z(t) = r(\text{συν}(t) + i\,\text{ημ}(t))\quad t \in [0, 2\pi]$

$f(z) = z + \frac{1}{z}$

$f(z(t)) = z(t) + \frac{1}{z(t)} = r(\text{συν}(t) + i\,\text{ημ}(t)) + \frac{1}{r(\text{συν}(t) + i\,\text{ημ}(t))}$

$= r(\text{συν}(t) + i\,\text{ημ}(t)) + \frac{1}{r}(\text{συν}(t) - i\,\text{ημ}(t))$

$= \left(r + \frac{1}{r}\right)\text{συν}(t) + i\left(r - \frac{1}{r}\right)\text{ημ}(t)$

$: \ w = u + iv$

Παραμετρικη παρασταση καμπυλης

$u = \left(r + \frac{1}{r}\right)\text{συν}t$

$t \in [0, 2\pi]\quad \frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{r^2}{\left(r-\frac{1}{r}\right)^2} = 1$

$v = \left(r - \frac{1}{r}\right)\text{ημ}t$

παριστα ΕΛΛΕΙΨΗ

Παραγώγιση / ολοκληρωσιμότητα

$$\varphi(h) = \frac{f(z_0+h) - f(z_0)}{h} = \frac{f(z) - f(z_0)}{z - z_0}$$

$$\exists \lim_{h\to 0} \varphi(h) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} =: f'(z_0)$$

$f$: $z = z_0 + h$ $h \neq 0$

$z_0 \in Z$

π.χ $f(z) = \bar{z}$

$$\frac{f(h) - f(0)}{h} = \frac{\bar{h}}{h}$$

$$\nexists \lim_{h\to 0} \left( \frac{f(h) - f(0)}{h} \right) = \frac{\bar{h}}{h}$$

δεν υπάρχει παράγωγο.

- $f$ ολόμορφη στο $z_0$: $\exists \delta > 0$: $\exists f'(z)\ \forall z \in B(z_0, \delta)$
- $f$ ακέραια ή αναλυτική:
  $\forall z \in \mathbb{C}$, $\exists f'(z)$